Swansea University
Mathematics scholarship exam 2022

2 hours 30 minutes

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $2\cdot(15)^\bullet$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 7/13 as a repeating decimal expansion.

2. Determine whether 667 is prime or not. Show your reasoning.

3. Solve the following simultaneous equations for x and y:

$$x - 3y = 2 \ , \quad 3x + y = 2 \ .$$

4. The number of bacteria in a culture increases by 7% every minute. If there were initially two million bacteria in the culture, how many would there be after 330 seconds?

5. Evaluate the integral

$$\int_0^2 \big(4x + 3\big)\ \mathrm{d}x \ .$$

6. There is one real number x which satisfies $x + 3x^4 = 1$ and $0 \le x \le 1$. Calculate x to within an error of $\frac{1}{16}$.

7. Find the differential with respect to x of

$$\frac{x^2+1}{x-1} \ .$$

8. Find numbers a, b, c, d so that

$$\frac{8x+4}{x^2+2x-3} = \frac{a}{x+b} + \frac{c}{x+d} \ .$$

9. A triangle has sides 4, 5 and 6. What is the size of the angle opposite the side of length 4?

10. Given that $x = -2$ is a root of the cubic $x^3 + 6x^2 + 4x - 8$, find the other two roots.

Section B

1. This question is about numbers of factors of integers – for example the integer 6 has four factors, which are 1,2,3,6.

a) How many factors has the number 15?

b) How many factors has the number $p\,q$, where p and q are different prime numbers? (The answer will not depend on the value of p or q.)

c) How many factors has the number $p^2\,q$, where p and q are different prime numbers?

d) How many factors has the number $p^n\,q^m$, where p and q are different prime numbers and n, m are positive integers?

e) What is the smallest positive integer which has exactly 9 factors.

2. With regard to a Cartesian frame of reference a particle of unit mass is projected upwards from the origin O at time $t = 0$, at an angle α to the horizontal, with speed v such that its position vector $\underline{\boldsymbol{r}}$ is always orthogonal to $\underline{\boldsymbol{j}}$. When the particle touches the x-axis it bounces upwards with speed $\frac{v}{2}$ again at an angle α to the horizontal. What is the position vector of the furthest point that the particle can travel and how for long does the motion persist?

3. Find the following infinite sums, where x is a real number in the interval $-1 < x < 1$.

$$a) \quad \sum_{n=1}^{\infty} \frac{x^n}{n} \ , \quad b) \quad \sum_{n=1}^{\infty} n\,x^n \ .$$

You may assume the following formula, but must explain your answers from there:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \ .$$

4. A can of beans consists of sheet metal in the shape of a right circular cylinder (i.e. the usual shape for a cylindrical can). Given that the radius of the circular base is r and the height of the can is h,
a) find the surface area of the can,
b) find the volume of the can.
We are given a total area of $2\,\pi$ square metres of sheet metal to make a large bean can.
c) Find a formula for h in terms of r.
d) What is the maximum volume can that we could make from this sheet metal?

5. A particle is constrained to move in the Cartesian plane, and travels from the origin $(0,0)$ to the point $(15,15)$ in the following manner;

a) the motion in the region $0 \le y \le 5$ is linear with the particle's speed being $v/4$

b) the motion in the region $5 \le y \le 15$ is linear with the particle's speed being v

If the particle crosses the line $y = 5$ at the point D, with coordinates $(d, 5)$;

(1) calculate the time, $t(d)$ taken for such a motion as a function of d,

(2) if $t(d)$ is minimised for $d = d_0$ obtain an equation satisfied by d_0,

(3) find an approximate value for d_0 by plotting $t'(d)$ in the interval $[1, 2]$.

6. A logician visits the island of Knights and Knaves. Remember that Knights always tell the truth, that Knaves always lie, and that every inhabitant is either a Knight or a Knave.

(a) On one occasion the logician meets two inhabitants A and B. He has no previous knowledge of whether either of the inhabitants is a Knight or a Knave. To find out something about this, he asks B the question 'is at least one of you a Knave?'. The reply from B is either 'Yes' or 'No', but because of the noise of a passing train you cannot hear which. However the logician did hear the reply, and remarks 'now I know exactly what A and B are'. Can you work out whether the answer was 'Yes' or 'No'? What are A and B?

(b) On another occasion the logician meets two inhabitants C and D while he is walking to the post office. The meeting takes place at a fork in the road, with two possible ways forward, left or right, and only one of these leads to the post office. The logician asks which way to go to get to the post office.

D says 'C is a Knight'.

C says 'if the way is left, then D is a Knave'.

Which way is the post office? Can you identify C and D?

7. The Gaulish chieftain Combinatorix is going to Rome to see Julius Cæsar, and is taking four druids and six warriors with him.

(a) On the journey to Rome the party walks in single file, with Combinatorix always at the front. Given that the warriors and druids can all be told apart, how many different ways can the party be arranged in a line?

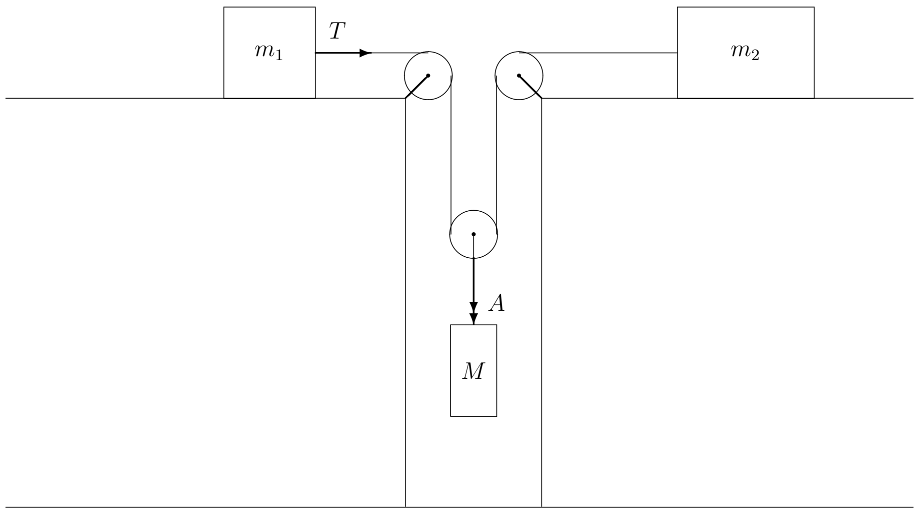
(b) They come to a village with two inns and stay for the night. Combinatorix takes the finest room, leaving the rest of the party to be split between the first inn, which has six places left, and the second inn, which has 4 places left. How many different ways can this splitting between the two inns be done?

(c) The next night they come to another village with two inns and stay for the night. Combinatorix again takes the finest room, leaving the rest of the party to be split between the first inn, which has six places left, and the second inn, which also has six places left. In how many different ways can this splitting between the two inns be done?

(d) The party stays at the village in (c) for a second night. The two innkeepers ask Combinatorix if he can arrange for exactly three warriors to stay in each inn, as they are afraid that too many warriors in one inn could cause trouble. How many different ways can this splitting between the two inns be done, given the restriction on placing the warriors?

(e) On leaving the village, the innkeepers present the four druids in the party with new robes. These robes are identical, and it is now impossible to tell the druids apart. The party again walks in single file, with Combinatorix leading the way, as they leave the village. In how many different distinguishable ways can the party be arranged in a line, given that all the druids now look the same?

8. Masses m_1 and m_2 sit on a smooth, (frictionless) horizontal table and are connected together by a light inextensible string. The string passes over two pulleys and under a third from which is suspended a mass M. The pulleys can be considered to be massless and rotate freely. Calculate the tension in the string T and the downward acceleration A of the mass M.



9. In a football league consisting of 4 teams, it is well known that Deeton has the following chances to win against the other teams:
Anystown $\frac{2}{3}$,
Beesville $\frac{1}{2}$,
Ceachester $\frac{1}{3}$.

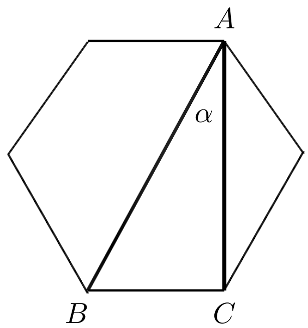
a) One year Deeton has to beat all the other sides in three games, one against each of the other teams. What is the probability that Deeton can win all three games?

b) The next year Deeton has to win at least one of three matches, one against each of the other teams. What is the probability that Deeton can win at least one of the three games?

c) The year after Deeton has to win one match against one of the other teams, where the opponent is chosen at random (probability $\frac{1}{3}$ that it could be any of the other teams). What is the probability that Deeton can win the game?

d) The year after that Deeton has to win two matches against two of the other teams (one match against each), where the side they do **not** play is chosen at random (probability $\frac{1}{3}$ that it could be any of the other teams). What is the probability that Deeton can win both the games?

10. The following figure is a regular hexagon, with side (e.g. BC) length 1.



Answer the following questions about the labelled vertices A, B, C. Explain your method.

a) What is the length of AC?

b) What is the length of AB?

c) What is the angle α of the triangle BAC?